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Note

An Optimum Time Step Length for Convergence to Steady-State Solution in Compressible-Flow Calculations

In making calculations on compressible convection in a closed box (two-dimensional problem for argon at 1 atm.) using the implicit factored scheme given by Beam and Warming [1], we have found that an optimum value of the time step length exists beyond which the rate of convergence to a steady state decreases rapidly. Using the 3-point-backward implicit scheme ($\theta = 1$, $\xi = 0.5$ in [1], no-slip boundary conditions at the walls, pressure at a wall derived from the equation for the momentum normal to the wall, implicit treatment of boundary conditions and source terms) and starting with zero-velocity initial values, it was observed that in a square box with side length $L = 5$ cm, the increase of the gas velocity, ΔV, in one time step increased linearly with the time step length, Δt, for values of $\Delta t < 10^{-5}$ sec (Courant number $C < 1$), levelled off to a maximum $(\Delta V)_{\max}$ at $\Delta t \approx 10^{-4}$ sec ($C \approx 10$) and decreased for still larger values of Δt, for $\Delta t > 10^{-3}$ sec about proportional to $(\Delta t)^{-1}$ (see Fig. 1, drawn lines). So the computed acceleration of the gas is large and independent of Δt for small values of Δt and represents correctly the physical process of gas acceleration. For large value of Δt the computed gas acceleration decreases proportional to $(\Delta t)^{-2}$ and does not longer represent the physical process. The decrease of the computed gas acceleration appeared to be a consequence of the decrease of the computed small negative pressure gradients driving the flow. The use of the trapezoidal implicit scheme ($\theta = 0.5$, $\xi = 0$ in [1]) gave slightly different results, see Fig. 1, dashed lines. By changing the side length L of the cavity and repeating the calculation, each time starting from the same zero-velocity initial values, it was found that $(\Delta V)_{\max}$ occurred at a value of the time step length $(\Delta t)_{\max}$ such that $(\Delta t)_{\max} \approx L/2A$, where A is the speed of sound in the gas; see Fig. 2. Changing the number of grid points of the uniform mesh at a constant value of L changed the Courant number at which $(\Delta V)_{\max}$ occurred but kept $(\Delta V)_{\max}$ itself at the same value of Δt; see Fig. 3.

As a consequence of the foregoing, the rate of convergence to a steady state, as expressed by the value of ΔV, decreased for Δt increasing beyond $(\Delta t)_{\max}$ for this two-dimensional problem. Such behaviour is also shown by the results obtained by Mahviladze *et al.* [2] for compressible convection. From the data given by them a relation $(\Delta t)_{\max} \approx L/2A$ can also be deduced. Thompkins *et al.* [3] report optimum values of the time step length in relation to convergence rate for inviscid supersonic problems.

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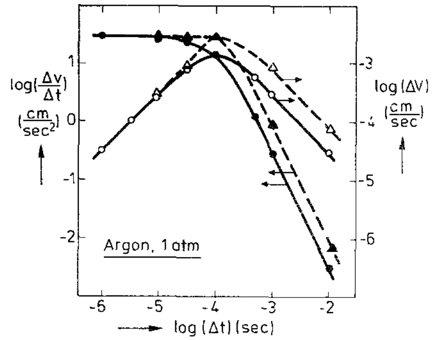

FIG. 1. Gas acceleration (●, ▲) and incremental change of gas velocity (○, △) as functions of time step length. Convection problem. Box side length $L = 5$ cm; $\Delta T = 30$ deg; number of space intervals: 17×17. 3-point-backward time differencing scheme (drawn lines) and trapezoidal time differencing scheme (dashed lines).

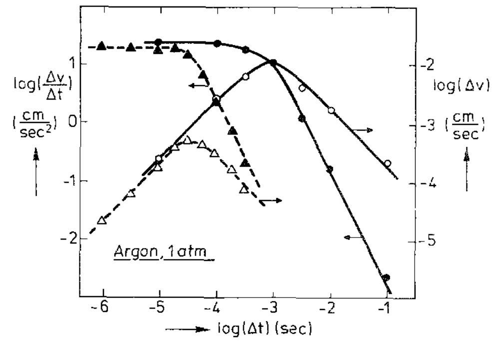

FIG. 2. As Fig. 1. Convection problem. Space intervals: 17×17. $L = 50$ cm, $\Delta T = 200$ deg (drawn lines); $L = 2.5$ cm, $\Delta T = 30$ deg (dashed lines).

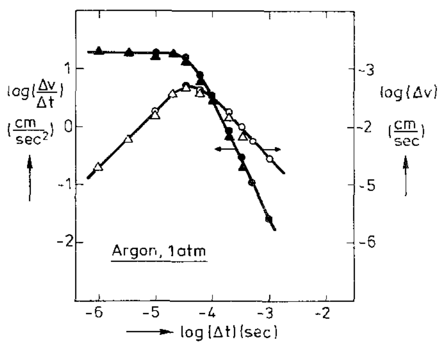

FIG. 3. As Fig. 1. Convection problem $L = 2.5$ cm, $\Delta T = 30$ deg. Space intervals: 9×9 (▲, △) and 17×17 (●, ○).

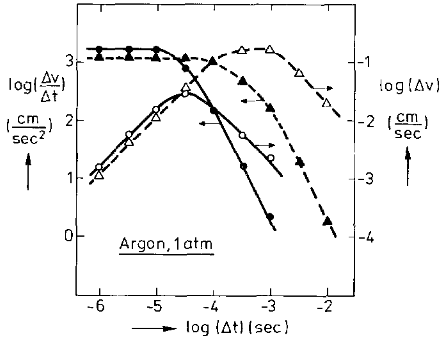

FIG. 4. As Fig. 1. Sliding-wall problem. Wall velocity: 5 cm/sec; space intervals: 17×17. 2.5×2.5 cm box (drawn lines); 2.5×50 cm box (dashed lines).

Analogous observations have been made by us in calculating the flow in a box due to a sliding motion of walls; see Fig. 4. From the results obtained here for a $(L_1 = 2.5\text{ cm}) \times (L_2 = 50\text{ cm})$ box it can be seen that the behaviour of the gas acceleration changes to $(\Delta t)^{-1}$-dependence in passing the point $\Delta t = L_1/2A$ and to $(\Delta t)^{-2}$-dependence in passing the point $\Delta t = L_2/2A$. In contrast to the convection problem, small positive pressure gradients built up, opposing the gas flow, when Δt increased beyond $(\Delta t)_{\max}$. For the one-dimensional problem obtained by letting $L_2 \to \infty$, the gas acceleration would have been proportional to $(\Delta t)^{-1}$ for all values of $\Delta t > L_1/2A$ and consequently ΔV, or the convergence rate, independent of Δt. This is in accordance with the remark of Thompkins *et al.* [3] in their conclusions that one-dimensional test examples showed no optimum convergence rate.

In order to demonstrate that the observed behaviour of the convergence rate is not caused by the non-linear or the viscous terms in the full Navier–Stokes equations, we solved the (constructed) system of equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0, \tag{1a}$$

$$\frac{\partial \rho u}{\partial t} + \frac{1}{m^2} \cdot \frac{\partial \rho}{\partial x} + \frac{GL_1}{u_0^2} \cdot \rho = 0, \tag{1b}$$

$$\frac{\partial \rho v}{\partial t} + \frac{1}{m^2} \cdot \frac{\partial \rho}{\partial y} = 0, \tag{1c}$$

in dimensionless quantities, where $m = u_0/A$. This system of equations is equivalent to the wave equation

$$\frac{\partial^2 \rho}{\partial t^2} - \frac{1}{m^2}\left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2}\right) = \frac{GL_1}{u_0^2} \cdot \frac{\partial \rho}{\partial x}. \tag{2}$$

Equations (1) have been solved, using again the Beam and Warming scheme, in a closed rectangular box ($L_1 \times L_2$ cm) with gravitation G applied over the full X-dimension of the box, however only over a part of the Y-dimension in order to drive the gas flow, starting always from zero-velocity initial values. The boundary conditions used in the Y-direction are

$$\left(\frac{\partial \rho}{\partial y}\right)_{\substack{y=0\\ y=L_2}} = 0, \qquad (\rho v)_{\substack{y=0\\ y=L_2}} = 0, \qquad \rho u\text{: extrapolated to the wall.}$$

For the X-direction the boundary conditions are

$$\left(\frac{\partial \rho}{\partial x}\right)_{\substack{x=0\\ x=L_1}} = 0, \qquad (\rho u)_{\substack{x=0\\ x=L_1}} = 0, \qquad \rho v\text{: extrapolated,}$$

for the region in which $G = 0$, and

$$\left(\frac{\partial \rho}{\partial x}\right)_{x=0} = c, \qquad \left(\frac{\partial \rho}{\partial x}\right)_{x=L_1} = c \cdot \exp(-G \cdot L_1/A^2),$$

$$(\rho u)_{\substack{x=0\\ x=L_1}} = 0, \qquad \rho v\text{: extrapolated,}$$

for the region in which $G \neq 0$, where

$$c = -(L_2/L_1) \cdot (G \cdot L_1/A^2)/\{1 - \exp(-G \cdot L_1/A^2)\}.$$

The behaviour of the computed gas acceleration as a function of time step length, box dimensions and number of mesh points is the same as described above for the full compressible Navier–Stokes equations.

The behaviour of the convergence rate as outlined above sets a rather severe upper bound to the time step length for compressible flow calculations using an implicit time differencing scheme. For $\Delta t < (\Delta t)_{\max}$ (covering also the time step domain in which explicit time differencing schemes operate) the computational time step length is equal to the physical time step length, a result of the direct method of solving the Navier–Stokes equations. For Δt increasing beyond $(\Delta t)_{\max}$, however, this connection between the computational and the physical time step length is gradually lost and the former more and more acquires the character of an iteration parameter. Therefore, Beam and Warming [1] state that for large Courant numbers the transient solutions deviate from the exact solution. Our results now indicate that the point from which the computational time step length starts to loose connection with the physical time step length is set by the propagation velocity of the waves represented by the partial differential equations together with the dimensions of the physical domain for which a steady state solution of these equations is sought. It will be clear that the widely used technique of increasing the time step length Δt when a steady state is approaching has no point if one is using already $\Delta t = (\Delta t)_{\max}$. Regarding CPU time consumption, taking account of the larger arithmetic operation count of implicit schemes, implicit finite difference calculations of compressible flows are thus advantageous with respect to explicit schemes only if the point $\Delta t = L/2A$ occurs at a Courant number $C \gg 10$.

Calculations of compressible flow in a closed box clearly reveal the effect of time step length on convergence rate, as outlined above. In calculations on compressible channel flow, a strongly decreased convergence rate, as will occur for time step lengths exceeding a problem-related upper bound as described above, may tempt one to think that the problem is approaching a steady state. Switching back to a much smaller time step length, however, then will reveal the truth.

References

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